Partitions of Integer Sets by Beatty Sequences

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1 Uniform Distribution of the Fractional Parts $i\alpha \mod 1$ and Complementary Beatty sequences

Given a positive irrational number $\alpha$ we can always find $\beta$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

thus meeting a well known criterion [2] for the sequences $S(\alpha) = \{\lfloor i\alpha \rfloor\}_{i=1}^\infty$ and $S(\beta) = \{\lfloor i\beta \rfloor\}_{i=1}^\infty$ to be complementary Beatty sequences, because they partition the positive integers. Of course, given $\beta$ we can also find such $\alpha$ and this relation can be expressed as

$$\beta = \frac{1}{1 - \frac{1}{\alpha}} = \frac{\alpha}{\alpha - 1},$$

(1)

which is an involution.

There are many proofs of this fact in the literature, but we cannot resist the temptation to point that it is immediate from Weyl’s well known result [7] about the uniform distribution of the fractional parts $i\gamma \mod 1$ in the interval $[0, 1)$ for any irrational $\gamma$, and from an incidental remark from Heilbronn found in a translation of a book by Vinogradov [6].

Indeed, an integer $m \in S(\gamma)$ if and only if

$$0 \leq 1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\},$$

(2)

where $\{x\}$ denotes the fractional part $x \mod 1$, because

1
\[ m = \left\lfloor \frac{m}{\gamma} \right\rfloor \gamma + \left\{ \frac{m}{\gamma} \right\} \gamma \]

\[ = \left\lfloor \frac{m}{\gamma} \right\rfloor \gamma + \gamma - \gamma + \left\{ \frac{m}{\gamma} \right\} \gamma \]

\[ = \left( \left\lfloor \frac{m}{\gamma} \right\rfloor + 1 \right) \gamma - \left( 1 - \left\{ \frac{m}{\gamma} \right\} \right) \gamma \]

Therefore, in order for \( m = \left\lfloor \left( \left\lfloor \frac{m}{\gamma} \right\rfloor + 1 \right) \gamma \right\rfloor \) (to be the floor of a multiple of \( \gamma \)) we need

\[ 0 \leq 1 - \left\{ \frac{m}{\gamma} \right\} \gamma < 1, \]

\[ 0 \leq 1 - \left\{ \frac{m}{\gamma} \right\} < \frac{1}{\gamma}, \]

\[ 0 \leq 1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\}, \]

which is the membership condition (2). Observe that the negation of this condition is just

\[ \frac{1}{\gamma} \leq 1 - \left\{ \frac{m}{\gamma} \right\}, \]

or, by applying the involution (1) in both sides of the inequality we get \( 1 - 1/(\gamma/(\gamma - 1)) \) in the left hand side, whereas the right hand side becomes

\[ 1 - \left\{ m(1 - \frac{1}{\gamma/(\gamma - 1)}) \right\} = 1 - \left\{ m - \frac{m}{\gamma/(\gamma - 1)} \right\}. \]

Now, this right hand side admits two cases:

**Case 1:** \( m > \frac{m}{\gamma/(\gamma - 1)} \)

\[ = 1 - \left\{ m - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} - \left\{ m \right\} \right\}, \]

\[ = 1 - \left\{ (k - 1) + 1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right\}, \]

where \( k \) is an integer, wherefore we get

\[ = 1 - \left\{ 1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right\}, \]

\[ = 1 - \left( 1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right), \]

\[ = \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\}. \]
Case 2: $m < \frac{m}{\gamma/(\gamma - 1)}$. Since the fractional part is now negative we subtract from 1,

$$= 1 - \left(1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right),$$

$$= 1 - \left(1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right) + \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} - m,$$

$$= 1 - \left(1 - \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \right),$$

$$= \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\}.$$

Putting it all together we get the membership condition for the complementary Beatty sequence

$$0 \leq 1 - \frac{1}{\gamma/(\gamma - 1)} \leq \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\}. \quad (3)$$

From conditions (2) and (3) and Weyl’s result we conclude that all integers appear in one of the complementary sequences and if a given integer is not found in $S(\gamma)$ then it must appear in $S(\gamma/(\gamma - 1))$, and viceversa. Also, recalling that Vinogradov[6] essentially established that the fractional parts $\{p\gamma\}$, where $p$ is prime and $\gamma$ irrational are also uniformly distributed in the unit interval, we see that the set of primes is likewise partitioned by complementary Beatty sequences, and that each set of any such partition contains an infinite number of primes. Thus, we have shown the following.

**Theorem 1** A pair of complementary Beatty sequences as defined above partition the set of positive integers.

**Idea of Proof:**

- All integers $m$ for which $1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\}$ are in $S(\gamma)$, whereas those for which $\left\{ \frac{m}{\gamma} \right\} \leq 1 - \frac{1}{\gamma}$ are in $S(\gamma/(\gamma - 1))$,

- for two integers $m \neq n$ we have $\left\{ \frac{m}{\gamma} \right\} \neq \left\{ \frac{n}{\gamma} \right\}$.

- Hence, an integer is either in $S(\gamma)$ or in the complementary sequence $S(\gamma/(\gamma - 1))$. QED

**Theorem 2** Each set of a partition of the integers defined by two complementary Beatty sequences contains an infinite number of integers.

**Idea of Proof:**
• H. Weyl [7], \(\{\frac{m}{\gamma}\}\) are equidistributed in \((0, 1)\).

• Hence, there are an infinity of fractional parts \(\{\frac{m}{\gamma}\}\) in the interval \((0, 1 - \frac{1}{\gamma})\), i.e., in \(S(\gamma)\), and likewise for \((1 - \frac{1}{\gamma}, 1)\) and \(S(\gamma/(\gamma - 1))\).

• Since each integer \(m\) maps to a distinct \(\{\frac{m}{\gamma}\}\), each sequence has an infinity of integers. QED

**Theorem 3** A pair of complementary Beatty sequences as defined above partition the set of primes.

**Idea of Proof:**
• All primes \(p\) for which \(1 - \frac{1}{\gamma} < \{\frac{p}{\gamma}\}\) are in \(S(\gamma)\), whereas those for which \(\{\frac{p}{\gamma}\} \leq 1 - \frac{1}{\gamma}\) are in \(S(\gamma/(\gamma - 1))\),

• for two primes \(p \neq q\) we have \(\{\frac{p}{\gamma}\} \neq \{\frac{q}{\gamma}\}\).

• Hence, an integer is either in \(S(\gamma)\) or in the complementary sequence \(S(\gamma/(\gamma - 1))\). QED

**Theorem 4** Each set of a partition of the integers defined by two complementary Beatty sequences contains an infinite number of primes.

**Idea of Proof:**
• I.M. Vinogradov [6], \(\{\frac{p}{\gamma}\}\) are equidistributed in \((0, 1)\).

• Hence, there are an infinity of fractional parts \(\{\frac{p}{\gamma}\}\) in the interval \((0, 1 - \frac{1}{\gamma})\), i.e., in \(S(\gamma)\), and likewise for \((1 - \frac{1}{\gamma}, 1)\) and \(S(\gamma/(\gamma - 1))\).

• Since each prime \(p\) maps to a distinct \(\{\frac{p}{\gamma}\}\), each sequence has an infinity of primes. QED

2 Symmetric partition of the Fibonacci sequence by Beatty sequences

In order to apply this line of thought to other interesting subsets of integers we need to recall a few well known facts from elementary number theory [3], [1].

We know that an irrational number \(\gamma\) can be represented as a continued fraction

\[
\gamma = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.
\]
which is written more succinctly as \( \gamma = [a_1, a_2, a_3, \ldots] \), where the \( a_i \) are positive integers. It is easy to prove by induction that if

\[
\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}
\]

(4)
then

\[
p_n = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}},
\]

where \( \frac{p_n}{q_n} \) is the \( n \)-th convergent of the continued fraction.

Post-multiplication of both sides by \( \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \) provides us with the following recurrent relations

\[
\begin{align*}
p_{n+1} &= a_{n+1}p_n + p_{n-1}, \\
q_{n+1} &= a_{n+1}q_n + q_{n-1}.
\end{align*}
\]

(5)

By taking determinants in both sides of equation (4) we see that

\[
p_nq_{n-1} - p_{n-1}q_n = (-1)^{n+1},
\]
from which it’s immediate that

\[
\frac{p_n}{q_n} = a_0 + \sum_{i=1}^{n} \frac{(-1)^{i-1}}{q_i q_{i-1}}.
\]

which leads us to observe that the convergents \( p_n/q_n \) bracket a decreasing interval alternately from each side, i.e., they follow this pattern:

\[
\frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \frac{p_{2m}}{q_{2m}} < \ldots < \frac{p_{2m-1}}{q_{2m-1}} < \ldots < \frac{p_1}{q_1}.
\]

Because the quantity \( \frac{1}{q_i q_{i-1}} \to 0 \) as \( i \to \infty \) is alternatively added to the left end of each new interval and subtracted from the right, the sequence of convergents has a limit

\[
\gamma = \lim_{n \to \infty} [a_0, a_1, \ldots, a_n],
\]

and the inequality \( |\gamma - \frac{p_i}{q_i}| \leq \frac{1}{q_i q_{i-1}} < \frac{1}{q_i^2} \) holds for all \( i > 0 \), the number \( \gamma \) must be irrational.

When the irrational number being approximated is the golden ratio \( \tau = (1 + \sqrt{5})/2 \), the coefficients of the continued fraction are all ones (which is seen by recursive application of the golden ratio defining property \( x = 1 + \frac{1}{x} \)). From
this fact and from the recursive relation (5) we conclude that the convergents
of this continued fraction are quotients of consecutive terms of the Fibonacci
sequence, i.e., \( F_n / F_{n-1} \).

Because the convergents \( F_n / F_{n-1} \) approximate \( \tau \) alternatively from left and
right, we have \( \frac{F_n}{\tau} < F_{n-1} \) for even \( n \), and \( \frac{F_n}{\tau} > F_{n-1} \) for odd \( n \), and since
\(|\tau - \frac{F_n}{F_{n-1}}| \to 0 \) as \( n \to \infty \) the fractional parts \( \{ \frac{F_n}{\tau} \} \) will tend monotonically
to 0 when \( n \) is odd and to 1 when \( n \) is even, and in either case these values
get away from \( 1 - \frac{1}{\tau} \), which marks the point on the interval \([0, 1)\) dividing the
subintervals that represent the membership conditions for each complementary
sequence. Then, by what we discussed in the beginning of the previous section
we have just proved that

**Theorem 5** The partition of the integers determined by the complementary
Beatty sequences \( S(\tau) \) and \( S(\tau/(\tau - 1)) \) contains all even indexed terms of the
Fibonacci sequence in the first partition and all odd indexed terms in the latter.

Since all Fibonacci primes have prime indexes in the Fibonacci sequence
except for \( F_4 = 3 \), the Beatty sequence \( S(\tau/(\tau - 1)) \) contains all Fibonacci
primes save that exception, so all Fibonacci primes (except one) have the form
\( p_F = \lfloor n \frac{\tau}{\tau - 1} \rfloor \), where \( \tau \) is the golden ratio, \( n \) is an integer and \( \frac{1}{\tau} \leq \left\{ \frac{p_F}{\tau/(\tau - 1)} \right\} < 1 \). The fractional parts

\[
\left\{ \frac{p_F}{\tau/(\tau - 1)} \right\} = \left\{ \frac{p_F}{\tau} - \frac{p_F}{\tau} \right\} = 1 - \left\{ \frac{p_F}{\tau} \right\} \quad (6)
\]

accumulate towards 1, while \( \left\{ \frac{p_F}{\tau} \right\} \) accumulates towards 0, because for particular
Fibonacci primes \( q_F < q'_F \), then \( \frac{1}{\tau} \leq \left\{ \frac{p_F}{\tau/(\tau - 1)} \right\} < \left\{ \frac{q_F}{\tau/(\tau - 1)} \right\} < 1 \).

Suppose that we choose a non-Fibonacci prime \( p \) from the sequence \( S(\tau/(\tau - 1)) \) and that it has the property \( \left\{ \frac{p_F}{\tau/(\tau - 1)} \right\} \in (1 - \epsilon, 1) \) for a small given \( \epsilon \), i.e.,
\( q/\tau < \epsilon \), that is, there is an integer \( q \) (also non-Fibonacci) such that
\( |\tau - \frac{q}{\tau}| < \epsilon_1 \), where \( \epsilon_1 \) depends on \( \epsilon \). In other words, \( \frac{q}{\tau} \) is a very good approximate of \( \tau \). However, it would not necessarily be a best approximate.

From a well known theorem of Lagrange we know that the convergents \( \frac{F_n}{F_{n-1}} \)
are best approximates in the sense that they provide the best approximation to
\( \tau \) considering the complexity (or height) of the denominator. In fact, [5] and [4]
state that the inequality

\[
|\tau - p/q| < \frac{1}{q^2},
\]

holds if and only if \( p \) and \( q \) are consecutive Fibonacci numbers.
Therefore, if there is a finite number of Fibonacci primes this would mean that there is a last Fibonacci prime $P_F$ after which there would be no prime number $p$ or $q$ such that the latter inequality holds. In other words, for integers greater than $P_F$ all rational best approximates of $\tau$ would have composite numerator and denominator.

References


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