A Diophantine Characterization of Fibonacci Primes

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Abstract

The main concern and contribution of this note is to find a characterization of the primes in the Fibonacci sequence that in combination with results on the distribution of the fractional parts \( p\alpha \mod 1 \), where \( p \) is a prime number and \( \alpha \) is an irrational, would shed light on the, as of today, still open question on their infiniteness.

A classic theorem from Legendre states that if \( p \) and \( q \) are relatively prime integers, with \( q > 0 \), where the following inequality holds,

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2},
\]

then \( \frac{p}{q} \) is a convergent of \( \alpha \). From this result and the well known fact that the convergents of the continued fraction of the irrational golden ratio \( \tau = \frac{1 - \sqrt{5}}{2} \) are fractions with denominator and numerator being consecutive terms of the Fibonacci sequence, one may immediately derive the following criterion after multiplication of the above inequality by \( q^2 \).

**Theorem 1** A given positive integer \( q \) is a member of the Fibonacci sequence if and only if

\[
q\|q\tau\| < \frac{1}{2},
\]

(1)

where \( \|x\| \) denotes the distance to the nearest integer to the real number \( x \).

There are other criteria for the Fibonacci numbers (see Simons [1]), but this is the simplest by far. The distance \( \|x\| \) may be defined in terms of \( \{x\} \), the fractional part of the real number \( x \), or \( x - \lfloor x \rfloor \). Then, criterion (1) looks as follows,

\[
q \left( \frac{1}{2} - \left| \{q\tau\} - \frac{1}{2} \right| \right) < \frac{1}{2},
\]

(2)
We are now ready to invoke an incidental remark from Heilbronn found in a translation of a book by Vinogradov [2], in the sense that the fractional parts \( \{p\alpha\} \), where \( p \) is a prime number and \( \alpha \) is an irrational, are uniformly distributed in the interval \((0, 1)\). We are interested in those primes which are also in the Fibonacci sequence, thus it is natural to apply Vinogradov’s result to inequality (2). Letting \( p \) run over the primes, \( \{pr\} \) maps them uniformly over the interval \((0, 1)\), and \( \|pr\| \) folds this map on the interval \((0, \frac{1}{2})\) (see Figure 1), hence these distances are also uniformly distributed (see Figures 2 and 4), but multiplication by the prime \( p \) produces values that are distributed in \((0, \frac{p_n}{2})\), where \( p_n \) is the \( n \)-th prime (Figure 5). Since different primes produce different values as \( n \to \infty \), their distribution mimics that of the primes, i.e., it tends to \( \pi(n) \), where \( \pi(x) \) is the number of primes smaller than or equal to \( x \) (see Figure 6).

There are an infinite number of primes, and the result of this mapping is densely distributed in such a way that values in the segment \((0, 0.5)\) occur with a positive probability, hence there are an infinite number of Fibonacci primes.

However, Fibonacci primes are incredibly slow to come by. There are only 7 in the first 10,000 primes, and 8 in the first 100,000.

References


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Figure 2: Nearest integer distances for the first 1000 primes.

Figure 3: Final mapping values for the first 1000 primes (after multiplying by $p$).
Figure 4: Histogram of the distances to nearest integer, first 1000 primes.

Figure 5: Histogram of final mapping values, for the first 100,000 primes.
Figure 6: Graph of $\frac{1}{\pi(x)}$