Distribution of Fibonacci Primes on a Pair of Complementary Beatty Sequences

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A. Fraenkel [1]: Given a positive irrational number $\alpha$ we can always find $\beta$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

The sequences

$$S(\alpha) = \{\lfloor i\alpha \rfloor \}_{i=1}^{\infty}$$

$$S(\beta) = \{\lfloor i\beta \rfloor \}_{i=1}^{\infty}$$

are called complementary Beatty sequences, because they partition the positive integers. We call $\alpha$ and $\beta$ conjugates of one another.
Complementary Beatty Sequences

\[ S(\tau) \]
\[ \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, 42, 43, 45, 46, 48, 50, 51, 53, 55, 56, 58, 59, 61, 63, 64, 66, 67, 69, 71, 72, 74, 76, 77, 79, 80, 82, 84, 85, 87, 88, 90, 92, 93, 95, 97, 98, 100, 101, 103, 105, 106, 108, 110, 111, 113, 114, 116, 118, 119, 121, 122, 124, 126, 127, 129, 131, 132, 134, 135, 137, 139, 140, 142, 144, 145, 147, 148, 150, 152, 153, 155, 156, 158, 160, 161 \} \\

\[ S(\frac{\tau}{\tau-1}) \]

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Necessary and sufficient condition

\[ m = \left\lfloor \frac{m}{\gamma} \right\rfloor \gamma + \left\{ \frac{m}{\gamma} \right\} \gamma \]

\[ = \left\lfloor \frac{m}{\gamma} \right\rfloor \gamma + \gamma - \gamma + \left\{ \frac{m}{\gamma} \right\} \gamma \]

\[ = \left( \left\lfloor \frac{m}{\gamma} \right\rfloor + 1 \right) \gamma - (1 - \left\{ \frac{m}{\gamma} \right\}) \gamma \]

therefore, in order for \( m = \left\lfloor (\left\lfloor \frac{m}{\gamma} \right\rfloor + 1) \gamma \right\rfloor \) (to be the floor of a multiple of \( \gamma \)) we need

\[ 0 \leq (1 - \left\{ \frac{m}{\gamma} \right\}) \gamma < 1, \]

\[ 0 \leq 1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\} \]
An integer $m \in S(\gamma)$ if and only if

\[ 0 \leq 1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\} \]

likewise, an integer $m \in S'(\gamma/(\gamma - 1))$ (the complementary sequence) if and only if

\[ 0 \leq 1 - \frac{1}{\gamma/(\gamma - 1)} \leq \left\{ \frac{m}{\gamma/(\gamma - 1)} \right\} \]

where $\{x\}$ is the fractional part $x \mod 1$. 

Salvador Gutiérrez, MAA Seaway 2010 – p. 5
Observe that conditions 2 and 3 can be represented as subintervals partitioning the interval $(0, 1)$, since

$$\frac{1}{\gamma} + \frac{1}{\gamma/(\gamma - 1)} = 1.$$
H. Weyl [5], “Über die Gleichverteilung von Zahlen mod. Eins.”

A sequence \( \{x_1, x_2, \ldots\} \) is equidistributed if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n<N} e^{2\pi i \, m \cdot x_n} = 0.
\]

As a consequence, the fractional parts \( \{n\alpha\} \) are equidistributed (hence dense) in \((0, 1)\) for \(\alpha\) irrational.
I.M. Vinogradov [4], “The Method of Trigonometrical Sums in the Theory of Numbers”

The fractional parts \( \{p\alpha\} \), where \( p \) is a prime and \( \alpha \) is an irrational number, are uniformly distributed (and dense) in the interval \((0, 1)\).
Immediate Consequences

Theorem 1 A pair of complementary Beatty sequences as defined above partition the set of positive integers.

Idea of Proof:

1. All integers \( m \) for which \( 1 - \frac{1}{\gamma} < \left\{ \frac{m}{\gamma} \right\} \) are in \( S(\gamma) \), whereas those for which \( \left\{ \frac{m}{\gamma} \right\} \leq 1 - \frac{1}{\gamma} \) are in \( S(\gamma/(\gamma - 1)) \).

2. For two integers \( m \neq n \) we have \( \left\{ \frac{m}{\gamma} \right\} \neq \left\{ \frac{n}{\gamma} \right\} \).

3. Hence, an integer is either in \( S(\gamma) \) or in the complementary sequence \( S(\gamma/(\gamma - 1)) \). QED
**Immediate Consequences**

**Theorem 2** Each set of a partition of the integers defined by two complementary Beatty sequences contains an infinite number of integers.

**Idea of Proof:**

1. H. Weyl [5], \( \{ \frac{m}{\gamma} \} \) are equidistributed in \((0, 1)\).

2. Hence, there are an infinity of fractional parts \( \{ \frac{m}{\gamma} \} \) in the interval \((0, 1 - \frac{1}{\gamma})\), i.e., in \(S(\gamma)\), and likewise for \((1 - \frac{1}{\gamma}, 1)\) and \(S(\gamma/(\gamma - 1))\).

3. Since each integer \(m\) maps to a distinct \( \{ \frac{m}{\gamma} \} \), each sequence has an infinity of integers. QED
**Immediate Consequences**

**Theorem 3** A pair of complementary Beatty sequences as defined above partition the set of primes.

**Idea of Proof:**

1. All primes \( p \) for which \( 1 - \frac{1}{\gamma} < \left\{ \frac{p}{\gamma} \right\} \) are in \( S(\gamma) \), whereas those for which \( \left\{ \frac{p}{\gamma} \right\} \leq 1 - \frac{1}{\gamma} \) are in \( S(\gamma/(\gamma - 1)) \),

2. for two primes \( p \neq q \) we have \( \left\{ \frac{p}{\gamma} \right\} \neq \left\{ \frac{q}{\gamma} \right\} \).

3. Hence, an integer is either in \( S(\gamma) \) or in the complementary sequence \( S(\gamma/(\gamma - 1)) \). QED
Theorem 4  Each set of a partition of the integers defined by two complementary Beatty sequences contains an infinite number of primes.

Idea of Proof:

I. M. Vinogradov [4], \( \left\{ \frac{p}{\gamma} \right\} \) are equidistributed in \((0, 1)\).

Hence, there are an infinity of fractional parts \( \left\{ \frac{p}{\gamma} \right\} \) in the interval \((0, 1 - \frac{1}{\gamma})\), i.e., in \( S(\gamma) \), and likewise for \((1 - \frac{1}{\gamma}, 1)\) and \( S(\gamma/(\gamma - 1)) \).

Since each prime \( p \) maps to a distinct \( \left\{ \frac{p}{\gamma} \right\} \), each sequence has an infinity of primes. QED
An irrational number $\gamma$ can be represented as a continued fraction

$$\gamma = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

which is written more succinctly as $\gamma = [a_1, a_2, a_3, \ldots]$, where the $a_i$ are positive integers.
It is easy to prove by induction that if

\[
\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}
\]

(4)

then

\[
\frac{p_n}{q_n} = a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}},
\]

where \( \frac{p_n}{q_n} \) is the \( n \)-th convergent of the continued fraction.
Post-multiplication of both sides by \( \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \) provides us with the following recurrent relations

\begin{align*}
 p_{n+1} &= a_{n+1}p_n + p_{n-1}, \\
 q_{n+1} &= a_{n+1}q_n + q_{n-1}.
\end{align*}

(5)
By taking determinants in both sides of equation (4) we see that

\[ p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \]

from which it’s immediate that

\[ \frac{p_n}{q_n} = a_0 + \sum_{i=1}^{n} \frac{(-1)^{i-1}}{q_i q_{i-1}}, \]
which leads us to observe that the convergents $\frac{p_n}{q_n}$ bracket a decreasing interval alternately from each side, i.e., they follow this pattern:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \frac{p_{2m}}{q_{2m}} < \ldots < \frac{p_{2m-1}}{q_{2m-1}} < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$
Because the quantity \( \frac{1}{q_i q_{i-1}} \to 0 \) as \( i \to \infty \) is alternatively added to the left end of each new interval and subtracted from the right, the sequence of convergents has a limit

\[
\gamma = \lim_{{n \to \infty}} [a_0, a_1, \ldots, a_n],
\]

and the inequality \( |\gamma - \frac{p_i}{q_i}| \leq \frac{1}{q_i q_{i-1}} < \frac{1}{q^2} \) holds for all \( i > 0 \), the number \( \gamma \) must be irrational.
Relation with the Fibonacci Numbers

When the irrational number being approximated is the golden ratio \( \tau = (1 + \sqrt{5})/2 \)

\[
\gamma = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}},
\]

(apply \( x = 1 + \frac{1}{x} \) recursively).

From this and from the recursive relation (5):

\[
p_{n+1} = a_{n+1}p_n + p_{n-1}
\]

We conclude that the convergents of this continued fraction are quotients of consecutive terms of the Fibonacci sequence, i.e., \( F_n/F_{n-1} \).
Relation with the Fibonacci Numbers

The convergents $F_n/F_{n-1}$ approximate $τ$ alternatively from left and right, so

\[ \frac{F_n}{τ} < F_{n-1} \quad \text{for even } n, \quad \text{and} \]

\[ \frac{F_n}{τ} > F_{n-1} \quad \text{for odd } n, \]

and since $|τ - \frac{F_n}{F_{n-1}}| \to 0$ as $n \to \infty$ the fractional parts

\( \{ \frac{F_n}{τ} \} \)

will tend monotonically to $0$ when $n$ is odd and to $1$ when $n$ is even.
Relation with the Fibonacci Numbers

In either case these values \( \{ \frac{F_n}{\tau} \} \) get away from \( 1 - \frac{1}{\tau} \). We have just proved that

**Theorem 5**  The partition of the integers determined by the complementary Beatty sequences \( S(\tau) \) and \( S(\tau/(\tau - 1)) \) contains all even indexed terms of the Fibonacci sequence in the first partition and all odd indexed terms in the latter.
All Fibonacci primes have prime indexes in the Fibonacci sequence except for $F_4 = 3$.

The Beatty sequence $S(\tau / (\tau - 1))$ contains all Fibonacci primes save that exception.

Hence, all Fibonacci primes (except one) have the form $p_F = \lfloor n \frac{\tau}{\tau - 1} \rfloor$, where $\tau$ is the golden ratio, $n$ is an integer and $\frac{1}{\tau} \leq \left\{ \frac{p_F}{\tau / (\tau - 1)} \right\} < 1$. 
The fractional parts

\[
\left\{ \frac{p_F}{\tau/(\tau - 1)} \right\} = \left\{ p_F - \frac{p_F}{\tau} \right\} = 1 - \left\{ \frac{p_F}{\tau} \right\}
\]

accumulate towards 1,

whereas the fractional parts \( \left\{ \frac{p_F}{\tau} \right\} \) accumulate towards 0.
Focus on the distribution of the fractional parts \( \left\{ \frac{p_F}{\tau/(\tau-1)} \right\} \) in the extreme right end of the interval \([0, 1)\).

From Vinogradov [4], we know that the fractional parts \( \left\{ \frac{p}{\tau/(\tau-1)} \right\} \), for \( p \) prime are uniformly distributed in the unit interval. Therefore, for arbitrary real \( \epsilon \) the probability that a particular arbitrary prime \( q \) will have the fractional part \( \left\{ \frac{q}{\tau/(\tau-1)} \right\} \) falling in the subinterval \((1 - \epsilon, 1)\) is proportional to \( \epsilon \).
Suppose that we choose a non-Fibonacci prime $p$ from the sequence $S(\tau/(\tau - 1))$ and that it has the property
\[
\left\{ \frac{p}{\tau/(\tau - 1)} \right\} \in (1 - \epsilon, 1)
\]
for a small given $\epsilon$, i.e., $\{p/\tau\} < \epsilon$, that is, there is an integer $q$ (also non-Fibonacci) such that $|\tau - \frac{p}{q}| < \epsilon_1$, where $\epsilon_1$ depends on $\epsilon$. In other words, $\frac{p}{q}$ is a very good approximate of $\tau$. However, it would not necessarily be a best approximate.
Fibonacci primes

From a well known theorem of Lagrange we know that the convergents $\frac{F_n}{F_{n-1}}$ are best approximates in the sense that they provide the best approximation to $\tau$ considering the complexity (or height) of the denominator. In fact, [3] and [2] state that the inequality

$$\left| \tau - \frac{p}{q} \right| < \frac{1}{q^2},$$

holds if and only if $p$ and $q$ are consecutive Fibonacci numbers.
Therefore, if there is a finite number of Fibonacci primes this would mean that there is a *last* Fibonacci prime $P_F$ after which there would be no prime number $p$ or $q$ such that the latter inequality holds. In other words, for integers greater than $P_F$ all rational best approximates of $\tau$ would have composite numerator and denominator.
Do we know enough about the occurrence of primes in the numerators or denominators of rationals approximating an irrational?
References


