Representation of Strict Closure Space Algebras

Jan Plaza
Computer Science Department
SUNY Plattsburgh

CUNY Graduate Center
Seminar in Logic and Games
October 28, 2011
Goldilocks’ thoughts on topology

- Closure spaces (Riesz, Moore, 1909-1910) – too weak
- Hausdorff spaces (Hausdorff, 1914) – too strong
- Topological spaces (Kuratowski, 1922) – just right
Definition of closure spaces

A **Closure space** – \( \langle X, Cl \rangle \), where \( X \) is a set

\( Cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) – **closure operation** s.t.:

\[
(\text{cls}_{Cl}) \quad B \subseteq ClB, \\
ClClB = ClB, \\
\text{If } B_1 \subseteq B_2 \text{ then } ClB_1 \subseteq ClB_2.
\]

\( \text{cls}_{Cl} \) is equivalent to: \( B_1 \subseteq ClB_2 \text{ iff } ClB_1 \subseteq ClB_2. \)

**Strict**, if also:

\[
(\text{str}_{Cl}) \quad Cl\emptyset = \emptyset.
\]

**Additive**, if also:

\[
(\text{add}_{Cl}) \quad ClA \cup ClB = Cl(A \cup B).
\]

**Topological space** = strict and additive closure space.
Examples of closure spaces

- Theories and a consequence operation (Tarski, 1930)
  - neither strict nor additive.
  Closed sets = theories closed under consequences.

- Binary relations and transitive closure
  - strict closure space.
  Closed sets = transitive relations.

- Binary relations and taking smallest extending congruence
  - neither strict nor additive.
  Closed sets = congruence relations.

- Closure of a set of database attributes under functional dependencies
  - strict closure space.
Alternative characterizations

Like topological spaces, closure spaces can be characterized using:

- Interior operation
- Family of closed subsets
- Family of open subsets
- Close base
- Open base
Interior operation

\[ \text{Int} : \mathcal{P}(X) \rightarrow \mathcal{P}(X), \text{ s.t.:} \]

\begin{align*}
\text{(cls}_{\text{Int}} & ) & & \text{Int}B \subseteq B \\
\text{Int} \text{Int}B & = \text{Int}B \\
\text{if } B_1 \subseteq B_2 \text{ then } \text{Int}B_1 & \subseteq \text{Int}B_2 \\
\text{(str}_{\text{Int}} & ) & & \text{Int}X = X \\
\text{(add}_{\text{Int}} & ) & & \text{Int}A \cap \text{Int}B = \text{Int}(A \cap B) \\
\end{align*}

Given an operation \( \text{Int} : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \), define \( \text{Cl}_{\text{Int}} : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) as

\[ \text{Cl}_{\text{Int}}A = X - \text{Int}(X - A). \]

Note: Replace \( \text{Int} \) by □ and these are axioms of S4.
Family of open sets

\[ \mathcal{O} \text{ s.t.:} \]

\[(cls_\mathcal{O}) \quad \emptyset \in \mathcal{O} \]
\[
\text{If } Q \subseteq \mathcal{O} \text{ then } \bigcup Q \in \mathcal{O}.
\]

\[(str_\mathcal{O}) \quad X \in \mathcal{O} \]

\[(add_\mathcal{O}) \quad \text{If } A, B \in \mathcal{O} \text{ then } A \cap B \in \mathcal{O} \]

Given a family \( \mathcal{O} \) of subsets of \( X \), define \( \text{Int}_\mathcal{O} : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) as

\[ \text{Int}_\mathcal{O}A = \bigcup\{B \in \mathcal{O} \mid B \subseteq A\}. \]
Open base

\( \mathcal{B} \subseteq \mathcal{P}(X) \) s.t.:

\begin{align*}
(cls_\mathcal{B}) & \quad \text{true} \\
(str_\mathcal{B}) & \quad \bigcup \mathcal{B} = X \\
(add_\mathcal{B}) & \quad \text{If } A, B \in \mathcal{B} \text{ then } \bigcup \{ C \in \mathcal{B} \mid C \subseteq A \cap B \} = A \cap B
\end{align*}

\( cls_\mathcal{B} \) means that any family of sets is an open base of a closure space.

Given a family \( \mathcal{B} \) of subsets of \( X \), define
\( \mathcal{O}_\mathcal{B} \subseteq \mathcal{P}(X) \) as
\( \mathcal{O}_\mathcal{B} = \{ \bigcup A \mid A \subseteq \mathcal{B} \}. \)

Notice that taking \( A = \emptyset \) gives \( \emptyset \in \mathcal{O}_\mathcal{B} \).

**Weight of** \( X \) – the smallest of cardinalities of open bases of \( X \).
Topological concepts

- **Open set**
- **Closed set**
- **Continuous function**
- **Open function** – where image of every open set is open
- **Dense subset**

Defined the same way as in topology.
From category theory

When continuous mappings are used as morphisms:

- **Homeomorphism**
- **Subspace**
- **Discrete space** – where every set is open
- **Power space** or cube $X^n$

$[\kappa]$ – discrete closure space of cardinality $\kappa$; notice that it is strict.
Powers

\[ X = \langle X, \mathcal{O} \rangle \] – a closure space.
\[ n \] – a cardinal.

Define \textit{(non-additive) power} or \textit{(non-additive) cube} \( X^n \) to be a closure space with:

- universe \( X^n \)

- open base \( \prod_{i<n'} X \times G \times \prod_{n'<i<n} X \) where \( n' < n \) and \( G \in \mathcal{O} \).

Note: if \( X \) is a topological space, its power in the category of topological spaces is not the same its power in the category of strict closure spaces.
Jankowski’s theorem (1985)

Let:

\[ Y_0 = \{g_1, g_2, f_1, f_2\} \] – strict closure space
with open sets family \( \mathcal{O} = \{\emptyset, \{g_1, g_2\}, Y_0\} \);
\( n \) – a cardinal number.

Then:

any strict closure space \( X \)
such that \( \text{card}(X) \leq 2^n \) and \( \text{weight}(X) \leq n \)
is homeomorphic to a subspace of the cube \( Y_0^n \).
Representation theorems

$X$ – a set, $P(X) = \langle P(X), \cap, \cup, \Rightarrow, -, X, \emptyset \rangle$.

**Boolean algebra of sets** – any subalgebra of $P(X)$.

$\mathcal{E}$ – all equations satisfied in all Boolean algebras of sets.

**Boolean algebra** – any $\langle A, \land, \lor, \rightarrow, -, \top, \bot \rangle$ that satisfies $\mathcal{E}$.

**Stone representation theorem** (1936)
Any Boolean algebra $A$ is isomorphic to a Boolean algebra of sets.
Representation theorems II

\[ X = \langle X, \text{Int} \rangle - \text{topological space} \]
\[ P(X) = \langle \mathcal{P}(X), \cap, \cup, \Rightarrow, -, X, \emptyset, \text{Int} \rangle \]

**Interior algebra of subsets of topological space**
– any subalgebra of \( P(X) \);

\( \mathcal{E} \) – all equations satisfied in all Boolean algebras of subsets of topological spaces.

**Interior algebra** – any \( \langle A, \land, \lor, \rightarrow, -, \top, \bot, \text{Int} \rangle \) that satisfies \( \mathcal{E} \); also known as **topological Boolean algebra**.

**Sikorski representation theorem** (1958)
For every countable interior algebra \( A \) there exists a set \( X_0 \) of irrational numbers s.t. \( A \) is isomorphic to an interior algebra of subsets of \( \langle X_0, \text{Int} \rangle \) with the topology inherited from the real line.
Representation vs. completeness
Algebra vs. logic

The same crucial lemma leads to representation and completeness:
... for every non-zero element of an algebra there exists a maximal
filter that contains that element ...

Rasiowa and Sikorski’s work (1950’s)

- Any countable set of infinite joins and meets in $\mathcal{A}$ can be
  preserved in the representation.

- Algebraic proof of Gödel’s completeness theorem.

- Completeness of a first-order version of modal logic S4.
Strict closure space algebras

\[ A = \langle A, \land, \lor, \rightarrow, -, Int, \top, \bot, \rangle \]

must satisfy:

1. The reduct \( \langle A, \land, \lor, \rightarrow, -, \top, \bot \rangle \) is a BA;

2. \( Int \) satisfies:
   \[
   \begin{align*}
   Inta & \leq a, \\
   Inta & \leq IntInta, \\
   \text{if } a \leq b \text{ then } Inta & \leq Intb;
   \end{align*}
   \]

3. \( Int \) also satisfies strictness condition:
   \[ Int \top = \top. \]

We define \( Cl \) as \( Cla = \neg Int - a. \)
Infinite joins and meets

A set $Q$ is called a set of infinite joins and meets in SCSA $A$ if:
\[
\langle \{a_i\}_{i \in I}, 0, a \rangle \in Q \implies \bigwedge_{i \in I} a_i = a,
\]
\[
\langle \{a_i\}_{i \in I}, 1, a \rangle \in Q \implies \bigvee_{i \in I} a_i = a.
\]

A homomorphism $f : A \rightarrow B$ is said to preserve $Q$ or to be a $Q$-homomorphism if:
\[
\langle \{a_i\}_{i \in I}, 0, a \rangle \in Q \implies \bigwedge_{i \in I} f(a_i) = f(a),
\]
\[
\langle \{a_i\}_{i \in I}, 1, a \rangle \in Q \implies \bigvee_{i \in I} f(a_i) = f(a).
\]

An isomorphic embedding $f$ of a SCSA $A$ into a complete SCSA $B$ is called a $Q$-isomorphic embedding if it preserves $Q$.

$BA$ $A$ is $Q$-representable if it can be $Q$-isomorphically embedded in $\mathcal{P}(X)$ for some set $X$. 

17
Universality of the Cantor cube

In topology, Cantor cube of weight $n$ ($[2]^n$) is universal for all zero-dimensional topological spaces (T1 spaces with an open base consisting of clopen sets) of weight $n$.

We will show analogous result for strict closure spaces.
Conditional Q-representation of SCSAs

$C(X)$ – the SCSA of all subsets of a strict closure space $X$.

**Lemma.**
Assume:
1. $n$ is an infinite cardinal;
2. $A$ is a SCSA of cardinality $\leq n$;
3. $Q$ is a set of infinite joins and meets from $A$.
Then:
if Boolean reduct of $A$ is $Q$-representable
then there exists a $Q$-isomorphic embedding of $A$ into SCSA $C(X)$ of all subsets of a strict closure space $X \subseteq [2]^n$. 
Sketch of the proof

1. There exists a $Q$-isomorphic embedding $f$ of $A$ into $C(X')$ where $X'$ is a certain strict closure space.

2. By Jankowski’s theorem there exists a homeomorphic embedding $h$ of $X'$ into $[Y_0]^n$.

3. Let $X'' = [2]^4$. There exists a decomposition of $X''$ into nonempty, disjoint sets $G_1, G_2, F_1, F_2$ such that $G_1 \cup G_2$ is an open set, $\text{Cl}G_1 = \text{Cl}G_2 = X''$, $\text{Cl}F_1 = \text{Cl}F_2 = X'' - G_1 - G_2$.

4. Using the decomposition from 3 we define a continuous, open mapping $\psi$ from $[\kappa]^n$ onto $[Y_0]^n$. 
5. The required $Q$-isomorphism can be constructed using the functions from 1, 2, 4:

\[ A \xrightarrow{Q\text{-iso emb}} C(X') \xleftarrow{Q\text{-iso}} C(\psi \vec{h}(X')) \]

Here $h^{-1}$ is the inverse function to $h$

$\vec{h}(\ldots)$ is the image of a set under $h$

$\psi (\ldots)$ is the inverse image under $\psi$.

We take $X = \psi(\vec{h}(X'))$.

QED
Theorem.
Assume:
1. $n$ is an infinite cardinal;
2. $A$ is a SCSA of cardinality $\leq n$;
3. $Q$ is at most countable set of infinite joins and meets from $A$.
Then:
there exists a $Q$-isomorphic embedding of $A$ into SCSA $C(X)$ of all subsets of a strict closure space $X \subseteq [2]^n$. 

Q-representation of SCSAs
Modal logic of strict closure spaces

Extend classical logic with schemas

N \( \Box \top, \)
T \( \Box \alpha \rightarrow \alpha, \)
4 \( \Box \alpha \rightarrow \Box \Box \alpha, \)
RM \( \alpha \rightarrow \beta / \Box \alpha \rightarrow \Box \beta. \)

SCS_0 – propositional
SCS_1 – first-order
No description operators, abstraction operators, lambda operator.
Lindenbaum algebras

$T$ – $\text{SCS}_1$-theory in language $L$, based on axioms $A$.
$n$ – infinite cardinal number.
$L'$ – the language obtained from $L$ by replacing its free individual variables by $x_\eta : \eta < n$.

$T'$ be an $\text{SCS}_1$-theory in $L'$ based on $A$.

Define $A_n(T)$ or the Lindenbaum algebra with $n$ variables, of $T$ – the quotient algebra of the algebra of formulas of $L'$ with respect to congruence:

$\alpha \sim \beta$ iff $\alpha \leftrightarrow \beta$ is a theorem of $T'$. 
Infinite joins and meets corresponding to quantifiers

If $n \geq \omega$ then

$$\bigcap_{t \in \text{Term}} \| \alpha(t) \| = \| (\forall \zeta) \alpha(\zeta) \|,$$

$$\bigcup_{t \in \text{Term}} \| \alpha(t) \| = \| (\exists \zeta) \alpha(\zeta) \|,$$

where $\text{Term}$ is the set of all terms of the language obtained by adding to $L$ the variables $x_\eta : \eta < n$.

The set of infinite joins and meets in $\mathcal{A}_n(T)$ which correspond to quantifiers is the set $Q$ containing

$$\langle \{\| \alpha(t) \| \}_t \in \text{Term}, 0, \| (\forall \zeta) \alpha(\zeta) \| \rangle,$$

$$\langle \{\| \alpha(t) \| \}_t \in \text{Term}, 1, \| (\exists \zeta) \alpha(\zeta) \| \rangle.$$

$Q$ does not contain all the infinite joins and meets from $\mathcal{A}_n(T)$: if $n > \omega$ then $\langle \{\| \alpha(x_\eta) \| \}_{\eta < \omega}, 0, \| (\forall \zeta) \alpha(\zeta) \| \rangle \notin Q$.

Notice that $\text{card}(Q) = \text{card}(\mathcal{A}_n(T))$. 
Q-filters, reducing meets and joins

Let $\mathcal{A}$ be a Boolean algebra with a set $\mathcal{Q}$ of infinite joins and meets, and let $\nabla$ be a proper filter in $\mathcal{A}$.

1. $\nabla$ is said to be a $\mathcal{Q}$-filter provided that the natural homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\nabla$ is a $\mathcal{Q}$-homomorphism, i.e. $\nabla = \overleftarrow{h}(\{\top\})$ for some $\mathcal{Q}$-homomorphism $h$. (Notice that unlike Rasiowa and Sikorski we do not require $\mathcal{Q}$-filters to be maximal.)

2. $\nabla$ is said to reduce the meets from $\mathcal{Q}$ provided that for all $\langle \{a_i\}_{i \in I}, 1, a \rangle$ and $\langle \{b_j\}_{j \in J}, 0, b \rangle$ from $\mathcal{Q}$:
   - if $\{b_j\}_{j \in J} \subseteq \nabla$ then $b \in \nabla$, and
   - if $\{-a_i\}_{i \in I} \subseteq \nabla$ then $-a \in \nabla$. 

26
Proposition.
Let $\nabla$ be a filter in a Boolean algebra with a set $Q$ of infinite joins and meets.

1. If $\nabla$ is a $Q$-filter then $\nabla$ reduces the meets from $Q$.

2. If $\nabla$ is a maximal filter then the following are equivalent:
   a) $\nabla$ is a $Q$-filter,
   b) $\nabla$ reduces the meets from $Q$. 
Proposition.
Assume:

\( T \) – first-order theory in a language of cardinality \( n \),
\( A_n(T) \) be the Lindenbaum algebra of \( T \) with \( n \) variables,
\( Q \) – the set of infinite joins and meets that correspond to quantifiers.

Then:

any filter \( \nabla \) in \( A_n(T) \)
generated by a set of cardinality less than \( n \)
reduces the meets of \( Q \).
Proof

Let $\nabla$ be generated by $\{\| \beta_i \| | i \in I\}$; $\text{card}(I) < \omega$.
Assume $\{\| \alpha(t) \| \}_{t \in \text{Term}} \subseteq \nabla$.
If $\| \alpha(t) \| \in \nabla$ then there exist formulas $\beta_{i_1}, \ldots, \beta_{i_k}$ such that
$\beta_{i_1} \land \ldots \land \beta_{i_k} \rightarrow \alpha(t)$ is a theorem of $T'$.
As there are $n$ variables among the terms in $\text{Term}$ and as the number of finite sequences
of $\beta$s is less than $n$, there must be a single sequence $\beta_{j_1}, \ldots, \beta_{j_m}$ and
an infinite set $V'$ of variables such that
for every $v \in V'$ we have $\beta_{j_1} \land \ldots \land \beta_{j_m} \rightarrow \alpha(v)$ is a theorem of $T'$.
As $V'$ is infinite,
there exists a variable $v' \in V'$ that does not occur in $\beta_{j_1} \land \ldots \land \beta_{j_m}$.
As $\beta_{j_1} \land \ldots \land \beta_{j_m} \rightarrow \alpha(v')$ is a theorem of $T'$,
also $\beta_{j_1} \land \ldots \land \beta_{j_m} \rightarrow (\forall \zeta) \alpha(\zeta)$ is a theorem of $T'$.
This implies that $\| (\forall \zeta) \alpha(\zeta) \| \in \nabla$.

QED
Q-representation of Lindenbaum algebras

Lemma.

Assume:
1. \( n \) is an arbitrary infinite cardinal;
2. \( L \) is a language of cardinality \( n \);
3. \( T \) is a \( \mathsf{SCS}_1 \) theory in \( L \);
4. \( Q \) is the set of infinite joins and meets in \( A_n(T) \) which correspond to quantifiers.

Then,
the Boolean reduct of the Lindenbaum algebra \( A_n(T) \) is \( Q \)-representable.

Note: it would not be true without \( \text{card} \, L \leq n \).
Sketch of the proof

Claim: for any non-zero element $\| \alpha \| \in \overline{A_n(T)}$ there exists a filter $\nabla^\alpha$ that contains $\| \alpha \|$, and is maximal, and is a $Q$-filter.

Let $\alpha_\eta : \eta < n$ be a sequence of all formulas.

Define:

$$G_0 = \left\{ \begin{array}{ll} \{\alpha, (\forall \zeta)\alpha_0(\zeta)\} & \text{if this set generates a proper filter.} \\ \{\alpha, \neg\alpha_0(v)\} & \text{where } v \notin \text{var}(\alpha), \text{ otherwise.} \end{array} \right.$$ 

$$\nabla_0 = \nabla(G_0)$$

$$G_\eta = \left\{ \begin{array}{ll} \bigcup_{\eta' < \eta} G_{\eta'} \cup \{(\forall \zeta)\alpha_\eta(\zeta)\} & \text{if this set generates a proper filter.} \\ \bigcup_{\eta' < \eta} G_{\eta'} \cup \{\neg\alpha_\eta(v)\} & \text{where } v \notin \text{var}(\bigcup_{\eta' < \eta} G_{\eta'}), \text{ otherwise.} \end{array} \right.$$ 

$$\nabla_\eta = \nabla(G_\eta).$$
\( \nabla_{\eta} : \eta < n \) are proper filters.

Each of them contains \( \| \alpha \| \).

Let \( \nabla^\alpha = \bigcup_{\eta < n} \nabla_{\eta} \).

\( \nabla^\alpha \) contains \( \| \alpha \| \).

\( \nabla^\alpha \) is a proper filter.

\( \nabla^\alpha \) is a maximal filter.

\( \nabla^\alpha \) reduces the meets from \( Q \).

So, \( \nabla^\alpha \) is a \( Q \)-filter.

So, for any non-zero element \( \| \alpha \| \in \overline{A_n(T)} \) we constructed a maximal filter that is a \( Q \)-filter and contains \( \| \alpha \| \).
Let $f'$ be the Stone isomorphic embedding of $\overline{A_n(T)}$ into Boolean algebra of all subsets of the set of all maximal filters in $\overline{A_n(T)}$.

Let $P_Q$ be the set of all maximal filters which are $Q$-filters in $\overline{A_n(T)}$.

We define $f : \overline{A_n(T)} \to \mathcal{P}(P_Q)$, $f(a) = f'(a) \cap P_Q$.
$f$ is an isomorphic embedding.

Let $X = P_Q$.

$f : \overline{A_n(T)} \to \mathcal{P}(X)$ is a $Q$-isomorphic embedding.

Notice that $\text{card}(X) \leq n$.

QED
Q-representation of Lindenbaum algebras

Assume that:

1. \( \kappa \) is a discrete strict closure space of cardinality \( \geq 2 \);

2. \( L \) is a language of cardinality \( n \);

3. \( T \) is an \( \text{SCS}_1 \) theory in \( L \);

4. \( Q \) is the set of infinite joins and meets in \( A_n(T) \) which correspond to quantifiers.

Then, there exists a \( Q \)-isomorphic embedding of \( A_n(T) \) into the SCSA \( C(X) \) of all subsets of a space \( X \subseteq [\kappa]^n \).