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LOGICS OF PUBLIC COMMUNICATIONS

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ABSTRACT.

Multimodal versions of propositional logics S5 or S4 - commonly accepted as logics of knowledge - are capable of describing static states of knowledge but they do not reflect how the knowledge changes after communications among agents. In the present paper (part of broader research on logics of knowledge and communications [10]) we define extensions of the logic S5 which can deal with public communications. The logics have natural semantics. We prove some completeness, decidability and interpretability results and formulate a general algorithm that solves certain kind of problems involving public communications - among them widely known puzzles of Muddy Children or Mr. Sum & Mr. Product.

As the paper gives formal logical treatment of the operation of restriction of the universe of a Kripke model, it contributes also to investigations of semantics for modal logics.

KEYWORDS: logics of knowledge, communications, Kripke models, logic S5, (applicable in) distributed systems, (applicable in) expert systems.

1 Introduction

A students' proverb says 'No one knows everything but true wisdom is to know whom to ask'. This emphasizes that knowledge can consist not only of ground facts but can also involve statements about somebody else's knowledge. Reasoning about knowledge is characteristic especially for the situations where information is exchanged.

Example 1.1 Muddy Children
Father : At least one of you has a muddy forehead.
Child 1 : I do not know whether my forehead is muddy.
Child 2 : I do not know whether my forehead is muddy.
Child 3 : I know whether my forehead is muddy or not but I won't tell you!
If the participants of the dialog can see each other (but no one can see his own forehead) is the forehead of Child 3 muddy?

Example 1.2 Mr. Sum & Mr. Product
Mr. Puzzle : I choose two natural numbers greater than 1. I will tell the sum of the
. numbers only to Mr. Sum and their product only to Mr. Product.
He tells them.
Mr. Product : I do not know the numbers.
Mr. Sum : I knew you didn't.
Mr. Product : But now I know!
Mr. Sum : So do I!
What can be the numbers if they are not greater than 100?

The first of the above puzzles was discussed by R. Parikh (cf. [9]), the second one is a classic exercise in combinatorics and was further popularized by J. McCarthy. In this paper we propose and investigate formal logical systems which can deal with communication sessions such as ones in 1.1, 1.2.
The communication session of Muddy Children consists of a sequence of four public communications. In the communication session of the example 1.2 Mr. Puzzle performs two semi-public communications - one directed to Mr. Product and one - to Mr. Sum. (Semi-public communications considered in [10] are beyond the scope of this paper and the results will be published elsewhere.) The first communication of Mr. Product and the first communication of Mr. Sum are both based only on the knowledge acquired after Mr. Puzzle's communications - Mr. Sum's communication does not depend on Mr. Product's one. Therefore we consider them as parallel (i.e. performed at the same time) public communications. After them two other public communications follow in a sequence.
Note that a communication consists not only of the message that was sent. To specify a communication one needs also description of the information received by various agents. Despite the fact that communication channels are guaranteed information received is usually

different from the message that was sent - its form depends on the kind of communication. For instance different information would be received by Mr. Sum if Mr. Puzzle sent the value of the sum of the numbers not in semi-public communication but in public one - Mr. Sum would know then that Mr. Product knows the sum.

Before we consider these problems in greater detail let us recall basic notions connected with logics of knowledge. (For general logical notions and those specific to modal logics the reader can consult [1], [3], S. Kripke's paper in [8] and [2].)

Definition 1.3 Language $\mathcal{L}_m(P)$
$\mathcal{L}_m(P)$ - the language of the logic of knowledge with m agents - is the language with a set P of propositional symbols and containing the following connectives:
$\wedge$, $\vee$, $\rightarrow$, $\equiv$, $\neg$, $\top$, $\bot$, K_1, …, K_m, Kw_1, …, Kw_m.
($\top$, $\bot$ stand for "true" and "false"; $\mathsf{K}_i\alpha$ can be read 'agent i knows that α is true',
$\mathsf{Kw}_i\alpha$ - 'agent i knows whether α is true or not'.)

Our basic tool is Kripke's possible worlds semantics. Each agent knows only some aspects of the situation that is considered. The agent does not know the actual world but he can imagine several possible worlds which do not differ in these aspects. Each group of such indistinguishable worlds constitutes an equivalence class of the indiscernibility relation associated with the agent. The agent knows a fact if the fact is true in all possible worlds which are indistinguishable from the actual world.

Definition 1.4 Kripke models
By a Kripke model (with equivalence relations) for a language $\mathcal{L}_m(P)$ we understand any tuple $M = \langle W, w_0, R_1, \ldots, R_m, v_P \rangle$, where:
W is a nonempty set (of possible worlds)
$w_0 \in W$ (is the actual world)
$R_1, \ldots, R_m \subseteq W \times W$ are equivalence relations (indiscernibility relations)
$v_P : W \times P \longrightarrow \{0, 1\}$ (is a valuation of propositional letters)
The relation of satisfaction of a formula in a world w of a Kripke model M is defined as the smallest relation meeting the following conditions:

$$\begin{array}{lll}
M, w \models_m \top & & \\
M, w \models_m p & \text{iff} & v_P(w, p) = 1 \;\; \text{(for any propositional letter } p \in P) \\
M, w \models_m \alpha \wedge \beta & \text{iff} & M, w \models_m \alpha \text{ and } M, w \models_m \beta \\
M, w \models_m \alpha \vee \beta & \text{iff} & M, w \models_m \alpha \text{ or } M, w \models_m \beta \\
M, w \models_m \alpha \rightarrow \beta & \text{iff} & M, w \models_m \alpha \text{ implies } M, w \models_m \beta \\
M, w \models_m \alpha \equiv \beta & \text{iff} & M, w \models_m \alpha \text{ is equivalent to } M, w \models_m \beta \\
M, w \models_m \neg \alpha & \text{iff} & \text{not } M, w \models_m \alpha \\
M, w \models_m \mathsf{K}_i \alpha & \text{iff} & \text{for any } w', \text{ if } w R_i w' \text{ then } M, w' \models_m \alpha \\
M, w \models_m \mathsf{Kw}_i \alpha & \text{iff} & M, w \models_m \mathsf{K}_i \alpha \text{ or } M, w \models_m \mathsf{K}_i \neg \alpha
\end{array}$$

A formula is said to be true in the model M iff $M, w_0 \models_m \alpha$; this is denoted by $M \models_m \alpha$. Given a set of formulas Γ we write $\Gamma \models_m \alpha$ to denote that α is true in all models in which formulas of Γ are true. A formula is universally true iff $\emptyset \models_m \alpha$, this is denoted by $\models_m \alpha$.

Definition 1.5 Logic $\mathbf{LK}_m$
For any language $\mathcal{L}_m(P)$ the consequence operation $\vdash_m$ of the logic of knowledge $\mathbf{LK}_m$ is defined by means of the following schemata ($i = 1, \ldots, m$):
1. axiom schemata of the propositional classical logic
2. $\mathsf{K}_i\alpha \to \alpha$
3. $\mathsf{K}_i\alpha \to \mathsf{K}_i\mathsf{K}_i\alpha$
4. $\neg\mathsf{K}_i\alpha \to \mathsf{K}_i\neg\mathsf{K}_i\alpha$
5. $\mathsf{K}_i\alpha \wedge \mathsf{K}_i(\alpha \to \beta) \to \mathsf{K}_i\beta$
6. $\mathsf{Kw}_i\alpha \equiv \mathsf{K}_i\alpha \vee \mathsf{K}_i\neg\alpha$
7. $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$
8. $\dfrac{\vdash_m \alpha}{\mathsf{K}_i\alpha}$

The logic $\mathbf{LK}_m$ can be viewed as a logic of an external observer who can reason about the world and about agent's knowledge. Even if the external observer can see that α is true it does not imply that the agents know α, therefore the rule $\frac{\alpha}{\mathsf{K}_i\alpha}$ is not assumed in $\mathbf{LK}_m$. We allow agents to be logically omniscient - to know all the logical theorems - ones which are true in all situations. Exactly this is expressed by the rule 8 - the assertion mark in the premiss of the rule indicates that it can be applied only to formulas which are logical theorems. Two examples: $\vdash_m \mathsf{K}_2(\mathsf{K}_1 p \to p)$ but $\{p\} \not\vdash_m \mathsf{K}_1 p$.

Remark 1.6
$\mathbf{LK}_m$ is a multimodal version of the Lewis logic **S5**.
$\mathbf{LK}_m$ is a conservative extension of the classical propositional logic.
Linguistical extensions of theories formalized in $\mathbf{LK}_m$ are conservative.
If $m < n$ then $\mathbf{LK}_n$ is a conservative extension of $\mathbf{LK}_m$:
if $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_m$ and $\Gamma \vdash_n \alpha$ then $\Gamma \vdash_m \alpha$.
Deduction lemma holds: $\Gamma \cup \{\alpha\} \vdash_m \beta$ iff $\Gamma \vdash_m \alpha \to \beta$.
$\mathbf{LK}_m$ is sound and complete: $\Gamma \models_m \alpha$ iff $\Gamma \vdash_m \alpha$.
$\mathbf{LK}_m$ is compact: if every finite subset of Γ has a model then Γ has a model.
$\mathbf{LK}_m$ has the finite model property (and therefore it is decidable):
if $\not\vdash_m \alpha$ then there exists a finite Kripke model M with equivalence relations such that $M \not\models_m \alpha$.
(cf. S. Kripke's paper in [8], [3], [6])

In example 1.2 Mr. Sum and Mr. Product speak about values of numbers. To express their statements we need new unary logical connectives Kv_i in our language. $\mathsf{Kv}_i d$ can be read

'agent i knows the value of the designator d'. For instance $\mathsf{Kv}_{Sum}\mathsf{numbers}$, $\mathsf{Kv}_{Holmes}\mathsf{murderer}$ or $\mathsf{Kv}_5\mathsf{temperature}$. The designators we consider are nonrigid - each can be thought as a name whose meaning varies from one world to another - for instance $\mathsf{temperature}$ can designate different real numbers in different possible worlds. Nonrigid designators are called sometimes nonrigid constants because they can be considered as individual constants of a first order language, constants - which have nonrigid semantical interpretations. (Yet most treatments of the first order modal logic interpret constants and functions as rigid. For a broader perspective cf. [11] , [2].) The language $\mathcal{L}^d_m(P,D)$ that is defined below is stronger than the propositional language $\mathcal{L}_m(P)$ but as it does not admit individual variables and quantifiers - it is still weaker than full first order modal language.
An agent is said to know the value of a designator d if d has the same value in all worlds indistinguishable from the actual one. Note that Kv_i can be thought as a generalization of Kw_i - in fact $\mathsf{Kw}_i\alpha$ means 'agent i knows the (logical) value of α'.

Definition 1.7 Language $\mathcal{L}^d_m(P,D)$
Consider a set P of propositional letters and a set D of individual constants. $\mathcal{L}^d_m(P,D)$ - the language of the logic of knowledge with designators - is the extension of the language $\mathcal{L}_m(P)$ in which for every $i = 1,\ldots,m$ and every $d \in D$ the expression $\mathsf{Kv}_i d$ is allowed as atomic formula.

Definition 1.8 Kripke models with nonrigid constants
By a Kripke model (with equivalence relations) and with nonrigid constants for a language $\mathcal{L}^d_m(P,D)$ we understand a tuple $M = \langle W, w_0, R_1, \ldots, R_m, v_P, v_D\rangle$
where $\langle W, w_0, R_1, \ldots, R_m, v_P\rangle$ is a Kripke model (with equivalence relations) for $\mathcal{L}_m(P)$ and v_D is a function with arbitrary range, defined on $W \times D$. The notion of satisfaction is defined as in usual Kripke models except that the following condition is added:
$M, w \models_m \mathsf{Kv}_i d$ iff for any w', w'', if wR_iw' *and* wR_iw'' then $v_D(w', d) = v_D(w'', d)$.

Note that that as the considered relations are equivalences we have
$w \models_m \mathsf{Kv}_i d$ iff for any w', if wR_iw' then $v_D(w', d) = v_D(w, d)$.

Definition 1.9 Logic $\mathbf{LK}^d_m$
For any language $\mathcal{L}^d_m(P,D)$ the consequence operation $\vdash^d_m$ of the logic of knowledge with nonrigid designato
$\mathbf{LK}^d_m$ is the extension of the logic $\mathbf{LK}_m$ obtained by adjoining the following schemata $(i = 1,\ldots,m)$:
1. $\mathsf{Kv}_i d \rightarrow \mathsf{K}_i\mathsf{Kv}_i d$
2. $\neg\mathsf{Kv}_i d \rightarrow \mathsf{K}_i\neg\mathsf{Kv}_i d$
3. $\dfrac{\vdash^d_m \alpha}{\mathsf{K}_i\alpha}$

The explanation following Definition 1.5 applies to rule 3 as well.

Remark 1.10
Linguistical extensions of theories formalized in $\mathbf{LK}^d_m$ are conservative.
$\mathbf{LK}^d_m(\mathcal{L}^d_m(P, D))$ is a conservative extension of $\mathbf{LK}_m(\mathcal{L}_m(P))$:
if $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_m(P)$ and $\Gamma \vdash^d_m \alpha$ then $\Gamma \vdash_m \alpha$.
If $m < n$ then $\mathbf{LK}^d_n$ is a conservative extension of $\mathbf{LK}^d_m$:
if $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^d_m$ and $\Gamma \vdash^d_n \alpha$ then $\Gamma \vdash^d_m \alpha$.
Deduction lemma holds: $\Gamma \cup \{\alpha\} \vdash^d_m \beta$ iff $\Gamma \vdash^d_m \alpha \rightarrow \beta$.
$\mathbf{LK}^d_m$is sound and complete: $\Gamma \vdash^d_m \alpha$ iff $\Gamma \models_m \alpha$.
$\mathbf{LK}^d_m$ is compact: if every finite subset of Γ has a model then Γ has a model.
$\mathbf{LK}^d_m$ has the finite model property and therefore it is decidable.
$\mathbf{LK}^d_m$ is interpretable in $\mathbf{LK}_m$, more exactly:
$\mathbf{LK}^d_m(\mathcal{L}^d_m(P, D))$ is interpretable in $\mathbf{LK}_m(\mathcal{L}_m(P \cup \{p_d : d \in D\}))$:
Let $^* : \mathcal{L}^d_m(P,D) \longrightarrow \mathcal{L}_m(P \cup \{p_d : d \in D\})$ be the mapping that replaces every occurrence of $\mathsf{Kv}_i d$ in a formula by $\mathsf{Kw}_i p_d$. Denote $\{\gamma^* : \gamma \in \Gamma\}$ by Γ^*. Then $\Gamma \vdash^d_m \alpha$ iff $\Gamma^* \vdash_m \alpha^*$.
cf. [10].

The mapping * above has the following informal meaning: instead of asking ‘Do you know who is the author of Knowledge and Belief ?’ ask ‘Do you know whether it is J. Hintikka who is the author of Knowledge and Belief ?’.

At the end of the preliminaries let us remind the concept of common knowledge. Define: $\mathsf{E}\alpha = \mathsf{K}_1\alpha \wedge \ldots \wedge \mathsf{K}_m\alpha$. $\mathsf{E}\alpha$ intuitively means ‘every agent knows that α is true’.
Define $\mathsf{E}^0\alpha = \alpha$, $\mathsf{E}^{n+1}\alpha = \mathsf{E}\mathsf{E}^n\alpha$. For instance $\mathsf{E}^2\alpha$ intuitively means ‘everybody knows that everybody knows that α is true’.
If the agents are gathered in a conference room and if somebody states aloud ground fact p then each agent gains infinite set of formulas: $\{\mathsf{E}^i p \ : \ i \in \mathcal{N}\}$ - in other words p becomes common knowledge. (Note that if not a ground fact p but a more complicated formula α were communicated it could happen that α would be no longer true after this communication; cf. Example 2.2.) Set $\{\mathsf{E}^i\alpha \ : i \in \mathcal{N}\}$ will be denoted $\mathsf{C}\alpha$ (C stands for ‘common’). Note that $\mathsf{C}\alpha$ represents an infinite conjunction and according to our terminology it is not a formula.
In a Kripke model $M = \langle W, w_0, R_1, \ldots, R_m, v_P \rangle$ (with equivalence relations) $M, w \models_m \mathsf{C}\alpha$ iff for any w', if $w R^{\mathsf{c}} w'$ then $M, w' \models_m \alpha$
where R^{c} is the transitive closure of the union $R_1 \cup \ldots \cup R_m$.
For a broader perspective on problems of reasoning about knowledge we recommend: J. Halpern's overview of the subject in [5], chapter 9 of [4] with Bibliographical and Historical Remarks 9.13, [6] - review of logics of knowledge, J Hintikka's classical book [7] and his paper in [8], and papers: [9], [5], [12].

2 Logics of public communications

We consider communication sessions with discrete time. The session begins at the time 0. A public communication can be imagined as a statement made in a conference room in which all agents are present. If at a moment t agent i starts a public communication with a message $\{\alpha\}$ then at moment $t+1$ it becomes common knowledge of all agents that i knew that α was true at t. (We consider only honest communications.) Although the situation involves time and common knowledge it can be described in a simpler way: the information received through this communication causes each agent to change the Kripke model he has at the time t - to delete possible worlds in which $\mathsf{K}_i\alpha$ is not true. This is the idea behind the following definition.

Definition 2.1
Let $\mathcal{L}_m^+$ be the extension of a language $\mathcal{L}_m(P)$ in which new binary logical connective $+$ is allowed. For any Kripke model $M = \langle W, w_0, R_1, \ldots, R_m, v_P\rangle$ for $\mathcal{L}_m(P)$ we define a notion of satisfaction of formulas of $\mathcal{L}_m^+$:
$M, w {\models_m} \alpha{+}\beta$ iff $M, w {\models_m} \alpha$ and $M \mid_\alpha, w {\models_m} \beta$
where $M \mid_\alpha$ is the restriction of the model M to the set $\{w \in W : M, w {\models_m} \alpha\}$.

Intuitively $\alpha{+}\beta$ is true iff β would be true after a (honest) public communication of $\{\alpha\}$ performed by an omniscient agent (i.e. an agent whose equivalence is equality).
Another example: β will be true in the situation after a sequence of (honest) public communications: $\{\alpha_1\}$ by the agent 1,..., of $\{\alpha_k\}$ by the agent k iff
$((\ldots(\mathsf{K}_1\alpha_1 + \mathsf{K}_2\alpha_2) + \ldots) + \mathsf{K}_k\alpha_k) + \beta$ is true at the present.
Another example: β will be true in the situation after (honest) parallel public communications of $\{\alpha_1\}$ by the agent 1 and of $\{\alpha_2\}$ by the agent 2 iff $(\mathsf{K}_1\alpha_1 \wedge \mathsf{K}_2\alpha_2) + \beta$ is true at the present.
More general:
For each i, t let Δ_i^t be a (honest) message sent in a public communication by the agent i at the time t. (Message is a finite set of formulas.) ($\Delta_i^t = \emptyset$ if there was no message.) Then β will be true at the time $t+1$ iff
$(\bigwedge_{i=1}^m \mathsf{K}_i\delta_i^0) + (\bigwedge_{i=1}^m \mathsf{K}_i\delta_i^1) + \ldots + (\bigwedge_{i=1}^m \mathsf{K}_i\delta_i^t) + \beta$ is true at the time 0, where $\delta_i^k = \bigwedge \Delta_i^k$ and $+$ is considered as right-associative. (Problem of placing parentheses in such expressions will disappear when we will learn in 2.14 that $+$ is associative.)

Example 2.2
Consider the language with the set of $P = \{p\}$ of propositional letters and a Kripke model $M = \langle W, w_0, R_1, R_2, v_P\rangle$ where
$W = \{w_0, w_1, w_2\}$,
equivalence classes of R_1 are $\{w_0\}$ and $\{w_1, w_2\}$,
equivalence classes of R_2 are $\{w_0, w_1\}$ and $\{w_2\}$,

$v_P(w_0, p) = 1$, $v_P(w_1, p) = 0$, $v_P(w_2, p) = 0$.
Consider $\alpha = \mathsf{K}_1\neg\mathsf{Kw}_2 p$. We will check whether the formula $\alpha \to \alpha + \alpha$ is true in w_0. We have: $M, w_0 \models_m \alpha$, $\quad M, w_1 \not\models_m \alpha$, $\quad M, w_2 \not\models_m \alpha$.
Let us consider the restricted model $M\mid_\alpha = \langle W\mid_\alpha, w_0, R_1\mid_\alpha, R_2\mid_\alpha, v_P\mid_\alpha\rangle$
Now $W\mid_\alpha = \{w_0\}$ and $R_1\mid_\alpha$, $R_2\mid_\alpha$ are equalities.
We have $M\mid_\alpha, w_0 \not\models_m \alpha$. Thus $M, w_0 \not\models_m \alpha{+}\alpha$. Consequently $M, w_0 \not\models_m \alpha \to \alpha + \alpha$.
This example shows a situation in which α is initially true in a model but after a public communication of $\{\alpha\}$, it becomes false.

Proposition 2.3
The following schemata are true in every Kripke model:
$\alpha{+}p \equiv \alpha{\wedge}p$ (for any propositional letter $p \in P$)
$\alpha{+}\top \equiv \alpha$
$\alpha{+}\bot \equiv \bot$
$\alpha{+}(\beta_1{\wedge}\beta_2) \equiv (\alpha{+}\beta_1){\wedge}(\alpha{+}\beta_2)$
$\alpha{+}(\beta_1{\vee}\beta_2) \equiv (\alpha{+}\beta_1){\vee}(\alpha{+}\beta_2)$
$\alpha{+}\neg\beta \equiv \alpha{\wedge}\neg(\alpha{+}\beta)$
$\alpha{+}(\beta_1{\to}\beta_2) \equiv \alpha{\wedge}(\alpha{+}\beta_1{\to}\alpha{+}\beta_2)$
$\alpha{+}(\beta_1{\equiv}\beta_2) \equiv \alpha{\wedge}(\alpha{+}\beta_1{\equiv}\alpha{+}\beta_2)$
$\alpha{+}\mathsf{K}_i\beta \equiv \alpha{\wedge}\mathsf{K}_i(\alpha \to \alpha{+}\beta)$

Definition 2.4 Logic $\mathbf{LK}_m^+$
For any language $\mathcal{L}_m^+$ the consequence operation $\vdash_m^+$ of the logic of public communications $\mathbf{LK}_m^+$ is defined as the extension of $\mathbf{LK}_m$ obtained by adding all the schemata listed in the proposition 2.3 and the rule of the replacement of equivalents: $\frac{\vdash_m^+ \alpha \equiv \beta}{\phi(\alpha) \equiv \phi(\beta)}$

The schemata of Proposition 2.3 are not independent. In fact it would be it would be enough to take as axioms of $\mathbf{LK}_m^+$ the first one and a subset of the remaining ones that corresponds to a complete set of logical connectives.

Theorem 2.5 Interpretability of $\mathbf{LK}_m^+$ in $\mathbf{LK}_m$
The equivalences of the proposition 2.3, schema 6 of Definition 1.5 and the rule of the replacement of equivalents determine a (unique) way of translating a formula α of $\mathcal{L}_m^+$ into a formula α^* of $\mathcal{L}_m$.
$*$ is an interpretation of $\mathbf{LK}_m^+$ in $\mathbf{LK}_m$: $\vdash_m^+ \alpha$ iff $\vdash_m \alpha^*$. Moreover $\vdash_m^+ \alpha \equiv \alpha^*$.

Proof.
The equivalence $\vdash_m^+ \alpha \equiv \alpha^*$ is a straightforward consequence of the definition of $*$.
By induction on ϕ show that $\models_m \alpha \equiv \beta$ implies $\models_m \phi(\alpha) \equiv \phi(\beta)$. This proves the soundness of $\mathbf{LK}_m^+$.

Now assume $\vdash_m \alpha^*$. Thus $\vdash^+_m \alpha^*$ and by the first equivalence: $\vdash^+_m \alpha$.
For the proof of the other implication assume $\vdash^+_m \alpha$. Thus by the first equivalence $\vdash^+_m \alpha^*$, thus by soundness of $\mathbf{LK}^+_m$: $\models_m \alpha^*$ and by completeness of $\mathbf{LK}_m$: $\vdash_m \alpha^*$.

Proposition 2.6
$+$ is not definable in terms of the remaining logical connectives:
there is no scheme $\phi(\alpha, \beta)$ such that: $+$ does not occur in ϕ and for any language $\mathcal{L}^+_m$ and any formulas in it $\vdash^+_m \alpha + \beta \equiv \phi(\alpha, \beta)$.

Proof.
If there were such a scheme ϕ then the rule of uniform substitution of formulas for propositional letters $\frac{\psi(p)}{\psi(\alpha)}$ would be admissible in $\mathbf{LK}^+_m$ (by induction on ψ). But as $\models_m \alpha + p \equiv \alpha \wedge p$ and $\not\models_m \alpha + \alpha \equiv \alpha \wedge \alpha$ (by Example 2.2) this rule is not valid. Contradiction.

Theorem 2.7 Soundness and completeness of $\mathbf{LK}^+_m$
$\Gamma \vdash^+_m \alpha$ iff $\Gamma \models_m \alpha$.

Proof.
Soundness was already mentioned in the proof of Theorem 2.5.
For the proof of completeness note that:
i) $\vdash^+_m \beta \equiv \beta^*$ (from 2.5)
ii) $\models_m \beta \equiv \beta^*$ (from i by soundness)
Assume $\Gamma \models_m \alpha$. Then by ii: $\Gamma^* \models_m \alpha^*$, thus by completeness of $\mathbf{LK}_m$: $\Gamma^* \vdash_m \alpha^*$, thus $\Gamma^* \vdash^+_m \alpha^*$, thus by i: $\Gamma \vdash^+_m \alpha$.

Proposition 2.8
If $m < n$ then $\mathbf{LK}^+_n$ is a conservative extension of $\mathbf{LK}^+_m$:
if $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^+_m$ and $\Gamma \vdash^+_n \alpha$ then $\Gamma \vdash^+_m \alpha$.

Proof.
Assume $\Gamma \not\vdash^+_m \alpha$. By completeness of $\mathbf{LK}^+_m$ there is a model M in which α is not true. M can be expanded to a model for $\mathbf{LK}^+_n$. By soundness: $\Gamma \not\vdash^+_n \alpha$.

Theorem 2.9 Compactness
$\mathbf{LK}^+_m$ is compact: if every finite subset of Γ has a model then Γ has a model.

Proof.
For the proof of compactness assume that $\Gamma \subseteq \mathcal{L}^+_m$ does not have a model. Then $\Gamma \models_m \bot$ and by completeness: $\Gamma \vdash^+_m \bot$. By finiteness of proofs there is finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash^+_m \bot$. Γ_0 does not have a model.

Theorem 2.10 On deduction
$\Gamma\cup\{\alpha\}\vdash^+_m\beta$ iff $\Gamma\vdash^+_m\alpha\rightarrow\beta$

Proof.
Assume $\Gamma\cup\{\alpha\}\vdash^+_m\beta$. By soundness: $\Gamma\cup\{\alpha\}\models_m\beta$. Thus $\Gamma^*\cup\{\alpha^*\}\models_m\beta^*$.
By completeness of $\mathbf{LK}_m$: $\Gamma^*\cup\{\alpha^*\}\vdash_m\beta^*$
and by deduction lemma for $\mathbf{LK}_m$: $\Gamma^*\vdash_m\alpha^*\rightarrow\beta^*$.
As $\alpha^*\rightarrow\beta^* = (\alpha\rightarrow\beta)^*$ we have $\Gamma^*\vdash_m(\alpha\rightarrow\beta)^*$, so: $\Gamma^*\vdash^+_m(\alpha\rightarrow\beta)^*$.
Therefore: $\Gamma\vdash^+_m\alpha\rightarrow\beta$.
The other implication in the theorem is straightforward.

Proposition 2.11
$\mathbf{LK}^+_m$ is a conservative extension of $\mathbf{LK}_m$: if $\Gamma\cup\{\alpha\}\subseteq\mathcal{L}_m$ and $\Gamma\vdash^+_m\alpha$ then $\Gamma\vdash_m\alpha$.

Proof.
Assume $\Gamma\vdash^+_m\alpha$. Thus by soundness of $\mathbf{LK}^+_m$: $\Gamma\models_m\alpha$, thus by completeness of $\mathbf{LK}_m$: $\Gamma\vdash_m\alpha$.

Theorem 2.12 Decidability of $\mathbf{LK}^+_m$
The logic $\mathbf{LK}^+_m$ has finite model property: if $\not\vdash^+_m\alpha$ then there exists a finite Kripke model M with equivalence relations such that $M\not\models_m\alpha$.
Therefore for any language $\mathcal{L}^+_m$ the relation $\vdash^+_m\alpha$ is decidable.

Note that in the above theorem α has to be a formula of a particular language, not a schema. We do not know how to decide whether a schema is admissible in $\mathbf{LK}^+_m$.

Proof.
Assume $\not\vdash^+_m\alpha$. Then by interpretability $\not\vdash_m\alpha^*$. By finite model property of $\mathbf{LK}_m$there exist required M such that $M\not\models_m\alpha^*$ and as $\models_m\alpha^*\equiv\alpha$ we have $M\not\models_m\alpha$.

A formula $\alpha\in\mathcal{L}^d_m(P,D)$ is said to be K-positive if it does not contain negative occurrences of K_i, Kw_i, Kv_i. In other words K-positive formulas are those equivalent to formulas built of classical formulas and formulas of the form $\mathsf{Kv}_i d$ by means of $\wedge$,$\vee$ and K_i.

Proposition 2.13
1. If a formula $\beta\in\mathcal{L}^+_m$ does not contain neither K_i nor Kw_i (but + is allowed) then $\vdash^+_m\alpha+\beta\equiv\alpha\wedge\beta$ and $\vdash^+_m\alpha+\mathsf{K}_i\beta\equiv\alpha\wedge\mathsf{K}_i(\alpha\rightarrow\beta)$.
2. If a formula $\beta\in\mathcal{L}^+_m$ is K-positive then $\vdash^+_m\alpha\wedge\beta\rightarrow\alpha+\beta$.

Proof.

1. By induction on β.
2. By the monotonic property of K-positive formulas:
   if γ is K-positive, $M, w \models_m \gamma, w \in M' \subseteq M$ then $M', w \models_m \gamma$.

Proposition 2.14

The following schemata are admissible in $\mathbf{LK}_m^+$:

$\alpha+(\beta+\gamma) \equiv (\alpha+\beta)+\gamma$

$\alpha+\beta \to \alpha$

$(\alpha_1 + \ldots + \alpha_i + \ldots + \alpha_n) \to (\alpha_1 + \ldots + \alpha_i)$

$\top+\alpha \equiv \alpha$

$\bot+\alpha \equiv \bot$

$(\alpha + \beta_1) \wedge (\alpha + (\beta_1 \to \beta_2)) \to (\alpha + \beta_2)$

$\alpha+\mathsf{K}_i\beta \to \alpha+\beta$

$\alpha+\mathsf{K}_i\beta \to \alpha+\mathsf{K}_i\mathsf{K}_i\beta$

$\alpha+\neg\mathsf{K}_i\beta \to \alpha+\mathsf{K}_i\neg\mathsf{K}_i\beta$

$\alpha + (\mathsf{K}_i\beta_1 \wedge \mathsf{K}_i\beta_2) \equiv \alpha + \mathsf{K}_i(\beta_1 \wedge \beta_2)$

$$\frac{\vdash^+_m \alpha \equiv \alpha',\ \vdash^+_m \beta \to \beta'}{\alpha+\beta \to \alpha'+\beta'}$$

$$\frac{\vdash^+_m \beta_1 \wedge \beta_2 \to \beta_3}{\alpha+\beta_1 \wedge \alpha+\beta_2 \to \alpha+\beta_3}$$

Proof.

All items can be proved semantically. We will show only the first item:
$M, w_0 \models_m \alpha + (\beta + \gamma)$ iff $M, w_0 \models_m \alpha$ and $M\,|_{\alpha}, w_0 \models_m \beta$ and $M\,|_{\alpha}|_{\beta}, w_0 \models_m \gamma$.
On the other hand
$M, w_0 \models_m (\alpha + \beta) + \gamma$ iff $M, w_0 \models_m \alpha$ and $M\,|_{\alpha}, w_0 \models_m \beta$ and $M\,|_{\alpha+\beta}, w_0 \models_m \gamma$.
It is enough to notice that $M\,|_{\alpha}|_{\beta} = M\,|_{\alpha+\beta}$ (but neither of those is $M\,|_{\alpha\wedge\beta}$).

Remark 2.15

The following schemata are **not** admissible in $\mathbf{LK}_m^+$:

$\alpha+\beta \equiv \beta+\alpha$

$\alpha+\beta \to \beta$

$\alpha \wedge \beta \to \alpha+\beta$

$\alpha \to \alpha+\alpha$

$\alpha + (\beta_1 + \beta_2) \equiv \alpha \wedge ((\alpha + \beta_1) + (\alpha + \beta_2))$

$$\frac{\psi(p)}{\psi(\alpha)}$$

$$\frac{\vdash^+_m \beta \to \beta'}{\alpha+\beta' \to \alpha+\beta}$$

$\dfrac{\vdash^{+}_{m}\alpha\to\alpha'}{\alpha+\beta\ \to\ \alpha'+\beta}$

$\dfrac{\vdash^{+}_{m}\alpha\to\alpha'}{\alpha'+\beta\ \to\ \alpha+\beta}$

Note also that although the rule $\dfrac{\vdash^{+}_{m}\beta\to\beta'}{\alpha+\beta\ \to\ \alpha+\beta'}$ is admissible,
$M\models_{m}\beta\to\beta'$ does not imply $M\models_{m}\alpha+\beta\to\alpha+\beta'$. (Compare also with 2.16.)

One of the counterexamples required for the proof of the above remark was given in 2.2, the remaining ones are left to the reader.
Immediately from the definition of semantics of + we obtain:

Proposition 2.16
If for any $w\in M$, $M,w\models_{m}\alpha\equiv\alpha'$ then for any $w\in M$, $M,w\models_{m}\alpha+\beta\equiv\alpha'+\beta$.

Proposition 2.16 is of importance for applications (cf. Example 2.17) and it should not be confused with the following statement which is **not** true:
if for any $w\in M$, $M,w\models_{m}\beta\equiv\beta'$ then for any $w\in M$, $M,w\models_{m}\alpha+\beta\equiv\alpha+\beta'$.
Also the following more general statement is **not** true:
if for any $w\in M$ $M,w\models_{m}\alpha\equiv\alpha'$ then for any $w\in M$ $M,w\models_{m}\phi(\alpha)\equiv\phi(\alpha')$

Example 2.17 Muddy Children II
Consider the model M corresponding to the initial situation in the puzzle of Muddy Children.
Possible worlds:
$W=\{\langle c,c,c\rangle,\langle c,c,m\rangle,\langle c,m,c\rangle,\langle c,m,m\rangle,\langle m,c,c\rangle,\langle m,c,m\rangle,\langle m,m,c\rangle,\langle m,m,m\rangle\}$
(Think of $\langle c,c,m\rangle$ as "Child1 is clean, Child2 is clean, Child3 is muddy".)
We do not specify the actual world in this model - we will test a formula corresponding to the dialog of Muddy Children in all the worlds of the model. If the formula is true in a world, the world can be taken as the actual world. In this way we will obtain all the worlds (situations) in which the dialog could take place.
Indiscernibility relations:
$\langle x_1,x_2,x_3\rangle R_{Father}\langle y_1,y_2,y_3\rangle$ iff $x_1=y_1, x_2=y_2$ and $x_3=y_3$
$\langle x_1,x_2,x_3\rangle R_{Child1}\langle y_1,y_2,y_3\rangle$ iff $x_2=y_2$ and $x_3=y_3$
$\langle x_1,x_2,x_3\rangle R_{Child2}\langle y_1,y_2,y_3\rangle$ iff $x_1=y_1$ and $x_3=y_3$
$\langle x_1,x_2,x_3\rangle R_{Child3}\langle y_1,y_2,y_3\rangle$ iff $x_1=y_1$ and $x_2=y_2$
(Intuitively two worlds look to Child1 alike if they agree on the second and on the third position - Child3 can see only the foreheads of Child2 and Child3, etc.)
In our language $\mathcal{L}^{+}_{4}$ we use the following propositional symbols:
atLeastOneMuddy, muddy1, muddy2, muddy3 with the following interpretations:
atLeastOneMuddy is true in the worlds containing at least one m,
muddy1 is true in the worlds represented by triples with an m at the first position,

$\mathsf{muddy2}$ is true in the worlds represented by triples with an m at the second position, $\mathsf{muddy3}$ is true in the worlds represented by triples with an m at the third position.
As explained after Definition 2.1 the formula corresponding to (sequential) dialog is created by prefixing the statement of agent i by K_i (for every agent) and joining the formulas obtained in this way by means of + (As + is associative at this time we do not bother with placing parentheses) :
$\mathsf{K}_{Father}\mathsf{atLeastOneMuddy}$ +
$+\mathsf{K}_{Child1}\neg\mathsf{Kw}_{Child1}\mathsf{muddy1} + \ \mathsf{K}_{Child2}\neg\mathsf{Kw}_{Child2}\mathsf{muddy2} + \ \mathsf{K}_{Child3}\mathsf{Kw}_{Child3}\mathsf{muddy3}$.
As $\vdash_m \mathsf{K}_i\neg\mathsf{Kw}_i\alpha \equiv \neg\mathsf{Kw}_i\alpha$, the formula is equivalent to:
$\mathsf{K}_{Father}\mathsf{atLeastOneMuddy} + \neg\mathsf{Kw}_{Child1}\mathsf{muddy1} + \neg\mathsf{Kw}_{Child2}\mathsf{muddy2} + \mathsf{Kw}_{Child3}\mathsf{muddy3}$
As for any $w \in W$, $\quad M, w \models_m \mathsf{K}_{Father}\mathsf{atLeastOneMuddy} \equiv \mathsf{atLeastOneMuddy}$
by proposition 2.16 it is enough to consider the following formula:
$\mathsf{atLeastOneMuddy} + \neg\mathsf{Kw}_{Child1}\mathsf{muddy1} + \neg\mathsf{Kw}_{Child2}\mathsf{muddy2} + \mathsf{Kw}_{Child3}\mathsf{muddy3}$.
One can see that there are exactly four worlds in which this formula is satisfied: $\langle c,c,m\rangle$, $\langle c,m,m\rangle$, $\langle m,c,m\rangle$, $\langle m,m,m\rangle$. One of them has to be the actual world of the agents. We do not have enough information to determine which one but we can see that all these worlds contain an m at the third position. Therefore the forehead of Child3 is muddy.

Remark 2.18
The operation + can be interpreted also in Kripke models with nonrigid constants. This leads to a (semantically defined) logic $\mathbf{LK}_m^{d+}$.
$\mathbf{LK}_m^{d+}$ is a conservative extension of both $\mathbf{LK}_m^{d}$ and $\mathbf{LK}_m^{+}$. (standard semantical proof that uses completeness of $\mathbf{LK}_m^{d}$ and $\mathbf{LK}_m^{+}$)
Proposition 2.13 generalizes to $\mathbf{LK}_m^{d+}$:
1. If a formula $\beta \in \mathcal{L}_m^{d+}$ does not contain K_i,Kw_i,Kv_i then $\vdash_m^{d+} \alpha + \beta \equiv \alpha \wedge \beta$.
2. If a formula $\beta \in \mathcal{L}_m^{d+}$ is K-positive then $\vdash_m^{d+} \alpha \wedge \beta \rightarrow \alpha + \beta$.
Moreover the following schemata are valid in $\mathbf{LK}_m^{d+}$:
$\mathsf{Kv}_i c + \mathsf{Kv}_i d \equiv \mathsf{Kv}_i c \wedge \mathsf{Kv}_i d$
$\mathsf{K}_i\alpha + \mathsf{Kv}_i d \equiv \mathsf{K}_i\alpha \wedge \mathsf{Kv}_i d$
$(\alpha + \mathsf{Kv}_i d) \rightarrow \mathsf{K}_i(\alpha \rightarrow (\alpha + \mathsf{Kv}_i d))$
$(\alpha + \neg\mathsf{Kv}_i d) \rightarrow \mathsf{K}_i(\alpha \rightarrow (\alpha + \neg\mathsf{Kv}_i d))$
$\dfrac{\models_m \alpha \equiv \beta}{\phi(\alpha) \equiv \phi(\beta)}$
We do not know whether the axioms of $\mathbf{LK}_m^{d}$ and $\mathbf{LK}_m^{+}$ augmented by the above ones give a complete axiomatization of $\mathbf{LK}_m^{d+}$.
Proposition 2.16 is valid also for Kripke models with nonrigid constants.

As the next example shows even without completeness $\mathbf{LK}_m^{d+}$ can be a useful tool.

Example 2.19 Mr. Sum & Mr. Product II
Consider the model M corresponding to the situation in the puzzle of Mr. Sum & Mr.

Product after Mr. Puzzle's communications.
Possible worlds: $W = \{\langle a,b\rangle \in \mathcal{N} \times \mathcal{N} \ : 1 < a \leq b\}$
Mr. Sum does not distinguish two worlds if they have the same sum, Mr. Product - if they have the same product:
$\langle a,b\rangle R_{Sum}\langle a',b'\rangle$ iff $a+b = a'+b'$,
$\langle a,b\rangle R_{Product}\langle a',b'\rangle$ iff $a*b = a'*b'$.
We do not specify the actual world in this model - we will test a formula corresponding to the dialog of Mr. Sum and Mr. Product in several worlds of the model. If the formula is true in a world, the world can be taken as actual world. In this way we will obtain all the worlds (situations) in which the dialog could take place.
Our language $\mathcal{L}_2^{d+}$ does not contain any propositional letters; it contains a nonrigid designator $\mathsf{numbers}$ which is interpreted in every possible world. Its value is the world itself: $v_D(w, \mathsf{numbers}) = w$ (for any $w \in W$).
As explained after Definition 2.1 the formula corresponding to the dialog is created by prefixing the statement of agent i by K_i (for every agent), joining parallel communications by means of $\wedge$ and sequential ones - by means of $+$. Note that the first statement of Mr. Product and the first statement of Mr. Sum are parallel communications, therefore the following formula corresponds to the dialog (As $+$ is associative we omit some parentheses):
$(\mathsf{K}_{Product}\neg\mathsf{Kv}_{Product}\mathsf{numbers} \wedge \mathsf{K}_{Sum}\neg\mathsf{Kv}_{Product}\mathsf{numbers})\ +$
$+\ \mathsf{K}_{Product}\mathsf{Kv}_{Product}\mathsf{numbers}\ +\ \mathsf{K}_{Sum}\mathsf{Kv}_{Sum}\mathsf{numbers}$
The following equivalences hold in $\mathbf{LK}_m^{d+}$ (in fact they are theorems of $\mathbf{LK}_m^{d}$) :
$(\mathsf{K}_{Product}\neg\mathsf{Kv}_{Product}\mathsf{numbers} \wedge \mathsf{K}_{Sum}\neg\mathsf{Kv}_{Product}\mathsf{numbers}) \equiv \mathsf{K}_{Sum}\neg\mathsf{Kv}_{Product}\mathsf{numbers}$,
$\mathsf{K}_{Product}\mathsf{Kv}_{Product}\mathsf{numbers} \equiv \mathsf{Kv}_{Product}\mathsf{numbers}$,
$\mathsf{K}_{Sum}\mathsf{Kv}_{Sum}\mathsf{numbers} \equiv \mathsf{Kv}_{Sum}\mathsf{numbers}$.
Therefore the formula representing dialog can be reduced to:
$\mathsf{K}_{Sum}\neg\mathsf{Kv}_{Product}\mathsf{numbers} + \mathsf{Kv}_{Product}\mathsf{numbers} + \mathsf{Kv}_{Sum}\mathsf{numbers}$
Because of the size of the problem it is better to employ a computer to test in which worlds $\langle a,b\rangle \in \mathcal{N} \times \mathcal{N} \ : 1 < a \leq b \leq 100$ the last formula is satisfied. (It is harmless that the model is infinite because equivalence classes of relations are always finite). The program returns four worlds: $\langle 4,13\rangle, \langle 4,61\rangle, \langle 16,73\rangle, \langle 64,73\rangle$.

We can consider another version of of the puzzle in which Mr. Puzzle tells Mr. Sum and Mr. Product in a public communication that the numbers are not greater than 100. Now the model is smaller: $W = \{\langle a,b\rangle \in \mathcal{N} \times \mathcal{N} \ : 1 < a \leq b \leq 100\}$. The formula representing dialog stays the same. The reader can write a PROLOG or LISP program and find the solution.

3 Concluding remarks

Because of applications in distributed systems and in expert systems logics of knowledge receive recently growing attention in computer science community. In both of these applications it is however essential to strengthen the expressive power of the logic to describe how the knowledge changes after communications among agents.

In this paper we discussed public communications and defined two corresponding logics allowing for two degrees of strength of the language. Although intuitive descriptions of public communications involve time and the notion of common knowledge we were able to eliminate them from our model. This elimination reduces the computational complexity of algorithms for testing satisfiability of formulas in possible worlds and makes them suitable for implementations.

The logics introduced in the paper can be used to solve in an automatic way problems similar to those of examples 1.1, 1.2.; in general - problems which satisfy the following assumptions:

- True knowledge: If an agent knows that α is true then it is true.
- Cumulative knowledge: Agents do not forget what they knew or heard.
- Honest messages: An agent communicates α only if he knows that α is true .
- Implicit knowledge discussed: Agents are perfect reasoners. For instance if an agent says 'I do not know α' it is not fault of his deductive abilities but α is not a logical consequence of his knowledge.
- Guaranteed communication channels: No message can be delivered late, misplaced, lost, changed or overheard.
- Common knowledge of external notions: For instance Mr. Sum and Mr. Product know in the sense of common knowledge what natural numbers are, what sum and product are.
- Messages expressed in a language of propositional logic of knowledge with nonrigid designators
- Public communication sessions: Agents start their session at the time 0, each with some primary knowledge. At any moment t any agent can initiate some communications (possibly many and possibly several agents at the same moment) sending messages based on the knowledge he has at this time. Messages are received at the time $t+1$ and contribute to the new states of agents' knowledge.
- Common initial Kripke model: Agents discuss an external object (world) which is fully characterized by values of its attributes. The agents know at the beginning of the session what combinations of values of attributes are possible. All these facts constitute common knowledge - Kripke models used by the agents have the same universe. Moreover if at the beginning of the session an agent knows values of some attributes then everybody knows (in the sense of common knowledge) that he knows the values of those attributes. So the kind of everybody's primary knowledge constitues common knowledge - agents know each other's indiscernibility relations in their initial Kripke model. To sum up - all the agents consider at the time $t = 0$ the same Kripke model.

- Complete description of communications: For instance if Mr. Sum knows eventually the numbers it is because of his reasoning and not because of a secret message sent to him by Mr. Puzzle, a message we do not know about.

While specifying the class of problems which can be solved using the logic of public communications we assumed that agents have the same initial model. In fact the above assumptions ensure that at each moment t agents have common Kripke model. Every next public communication changes this model but the new model is also common to all the agents.

Algorithm

Given a problem satisfying the bulleted assumptions above, define the corresponding Kripke model, express the dialog of the agents by a formula of $\mathbf{LK}_m^+$ or $\mathbf{LK}_m^{d+}$, simplify the formula using equivalences of the logic or Proposition 2.16 and test in which worlds of the model it is true.

A test whether a formula of $\mathbf{LK}_m^{d+}$ is satisfied in a world of a given Kripke model can be implemented in PROLOG in a very natural way. The universe of the model can be represented by a procedure `worlds(PossibleWorld)` which generates under backtracking all possible worlds. Indiscernibility relations can be represented by a procedure `relation(Agent, World, AccessibleWorld)` which generates under backtracking all the worlds indiscernible from the given one. (If equivalence classes of relations in our model are infinite the program need not terminate for some queries.) Propositional letters can be represented by unary tests which return `Yes` if they are true in a world. Nonrigid designators can be represented as procedures which take a world as their first argument and return the (unique) value of the designator on the second argument. PROLOG's device of infix and prefix operators allows us to write formulas in a transparent way. The procedure `satisfied(World, Formula, YesOrNo)` can be defined recursively according to the definitions 1.4, 1.8 and + can be translated away by the mapping * from Theorem 2.5
Due to its associativity there are multiple ways of translating + away from a formula. They lead to equivalent but not identical formulas, for instance:
$p + q + r \equiv p + (q + r) \equiv p + (q \wedge r) \equiv (p + q) \wedge (p + r) \equiv (p \wedge q) \wedge (p + r) \equiv (p \wedge q) \wedge (p \wedge r)$
$p + q + r \equiv (p + q) + r \equiv (p + q) \wedge r \equiv (p \wedge q) \wedge r$
As we see, defining + as left associative saves some work.
The program based on above ideas is purely recursive and it does not use much memory but it possibly repeats the same calculation several times.
Another way of handling + can be based on Definition 2.1. It leads to an iterative program which uses a lot of memory (especially for Kripke models with big universes) and possibly carries out some computations which are irrelevant to a problem (but does each such calculation only once).
The most reasonable choice for implementation is an algorithm intermediate between the

mentioned approaches - one which uses recursion but stores obtained results to avoid repeating the calculations.
Illustrating programs can be obtained from the author via E-mail.

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